Lecture 35

Spectral Expansions of Source Fields

In previous lectures, we have assumed plane waves in finding closed form solutions. Plane waves are simple waves, and their reflections off a flat surface or a planarly layered medium can be found easily. When we have a source like a point source, it generates a spherical wave. We do not know how to reflect exactly a spherical wave off a planar interface. But by expanding a spherical wave in terms of sum of plane waves and evanescennt waves using Fourier transform technique, we can solve for the solution of a point source near a layered medium easily in terms of spectral integrals. Sommerfeld was the first person to have done this, and hence, these integrals are often called Sommerfeld integrals. Finally, we shall apply the method of stationary phase to approximate these integrals to elucidate their physics. From this, we can see ray theory emerging from the complicated mathematics. It reminds me of a lyric from the musical The Sound of Music—Ray, a drop of golden sun! Ray has mesmerized the human mind, and it will be interesting to see if the mathematics behind it is equally enchanting.

35.1 Spectral Representations of Sources

A plane wave is a mathematical idealization that does not exist in the real world. In practice, waves are nonplanar in nature as they are generated by finite sources, such as antennas and scatterers. For example, a point source generates a spherical wave which is nonplanar. Fortunately, these waves can be expanded in terms of sum of plane waves. Once this is done, then the study of non-plane-wave reflections from a layered medium becomes routine. In the following, we shall show how waves resulting from a point source can be expanded in terms of plane waves summation. This topic is found in many textbooks [1,31,34,88,89,167,191,204].

35.1.1 A Point Source

From this point onward, we will adopt the $\exp(-i\omega t)$ time convention to be commensurate with the optics and physics literatures.

There are a number of ways to derive the plane wave expansion of a point source. We will illustrate one of the ways. The spectral decomposition or the plane-wave expansion of the field due to a point source could be derived using Fourier transform technique. First, notice that the scalar wave equation with a point source is

$$\left(\nabla^2 + k_0^2\right)\phi(x, y, z) = \left[\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} + k_0^2\right]\phi(x, y, z) = -\delta(x)\,\delta(y)\,\delta(z). \tag{35.1.1}$$

The above equation could then be solved in the spherical coordinates, yielding the solution given in the previous lecture, namely, Green's function with the source point at the origin, or

$$\phi(x, y, z) = \phi(r) = \frac{e^{ik_0 r}}{4\pi r}. \tag{35.1.2}$$

The solution is entirely spherically symmetric due to the symmetry of the point source.

Next, assuming that the Fourier transform of $\phi(x, y, z)$ exists,[1] we can write

$$\phi(x, y, z) = \frac{1}{(2\pi)^3}\iiint\limits_{-\infty}^{\infty} dk_x dk_y dk_z\, \tilde{\phi}(k_x, k_y, k_z) e^{ik_x x + ik_y y + ik_z z}. \tag{35.1.3}$$

Then we substitute the above into (35.1.1), after exchanging the order of differentiation and integration, one can simplify the Laplacian operator in the Fourier space, or spectral domain, to arrive at

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} = -k_x^2 - k_y^2 - k_z^2$$

Then, together with the Fourier representation of the delta function, which is

$$\delta(x)\,\delta(y)\,\delta(z) = \frac{1}{(2\pi)^3}\iiint\limits_{-\infty}^{\infty} dk_x dk_y dk_z\, e^{ik_x x + ik_y y + ik_z z} \tag{35.1.4}$$

we convert (35.1.1) into[2]

$$\iiint\limits_{-\infty}^{\infty} dk_x dk_y dk_z\, [k_0^2 - k_x^2 - k_y^2 - k_z^2]\tilde{\phi}(k_x, k_y, k_z) e^{ik_x x + ik_y y + ik_z z} \tag{35.1.5}$$

$$= -\iiint\limits_{-\infty}^{\infty} dk_x dk_y dk_z\, e^{ik_x x + ik_y y + ik_z z}. \tag{35.1.6}$$

[1]The Fourier transform of a function $f(x)$ exists if it is absolutely integrable, namely that $\int_{-\infty}^{\infty}|f(x)|dx$ is finite (see [203]).

[2]We have made use of that $\delta(x) = 1/(2\pi)\int_{-\infty}^{\infty} dk_x \exp(ik_x x)$ three times.

Since the above is equal for all x, y, and z, we can Fourier inverse transform the above to get

$$\tilde{\phi}(k_x, k_y, k_z) = \frac{-1}{k_0^2 - k_x^2 - k_y^2 - k_z^2}. \tag{35.1.7}$$

Consequently, we have

$$\phi(x, y, z) = \frac{-1}{(2\pi)^3} \iiint_{-\infty}^{\infty} d\mathbf{k} \, \frac{e^{ik_x x + ik_y y + ik_z z}}{k_0^2 - k_x^2 - k_y^2 - k_z^2}. \tag{35.1.8}$$

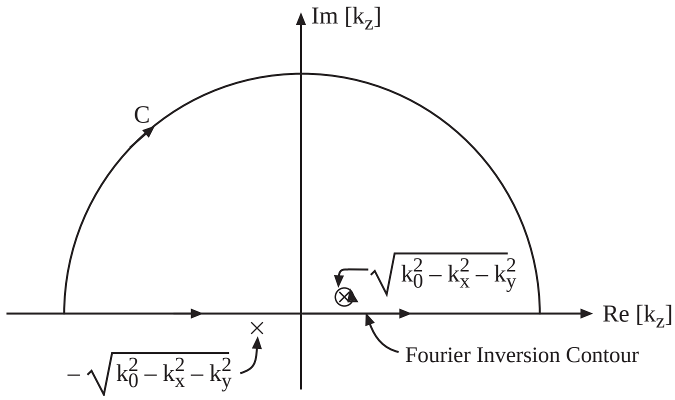

Figure 35.1: The integration along the real axis is equal to the integration along C plus the residue of the pole at $(k_0^2 - k_x^2 - k_y^2)^{1/2}$, by invoking Jordan's lemma.

Weyl Identity

In the above, if we examine the k_z integral first, then the integrand has poles at $k_z = \pm(k_0^2 - k_x^2 - k_y^2)^{1/2}$.[3] Moreover, for real k_0, and real values of k_x and k_y, these two poles lie on the real axis, rendering the integral in (35.1.8) undefined. However, if a small loss is assumed in k_0 such that $k_0 = k_0' + ik_0''$, then the poles are off the real axis (see Figure 35.1), and the integrals in (35.1.8) are well-defined. As we shall see, this is intimately related to the uniqueness principle we have studied before: An infinitesimal loss is needed to guarantee uniqueness in an open space.

First, the reason is that $\phi(x, y, z)$ is not strictly absolutely integrable for a lossless medium, and hence, its Fourier transform may not exist [47]. Second, the introduction of a small loss also guarantees the radiation condition and the uniqueness of the solution to (35.1.1), and therefore, the equality of (35.1.2) and (35.1.8) [34].

Observe that in (35.1.8), when $z > 0$, the integrand is exponentially small when $\Im m[k_z] \to \infty$. Therefore, by Jordan's lemma, the integration for k_z over the contour C as shown in Figure

[3]In (35.1.8), the pole is located at $k_x^2 + k_y^2 + k_z^2 = k_0^2$. This equation describes a sphere in $\mathbf{k}$ space, known as the Ewald's sphere [205].

35.1 vanishes. Then, by Cauchy's theorem, the integration over the Fourier inversion contour on the real axis is the same as integrating over the pole singularity located at $(k_0^2 - k_x^2 - k_y^2)^{1/2}$, yielding the residue of the pole (see Figure 35.1). Consequently, after doing the residue evaluation, we have

$$\phi(x, y, z) = \frac{i}{2(2\pi)^2} \iint_{-\infty}^{\infty} dk_x dk_y \, \frac{e^{ik_x x + ik_y y + ik_z' z}}{k_z'}, \quad z > 0, \tag{35.1.9}$$

where $k_z' = (k_0^2 - k_x^2 - k_y^2)^{1/2}$.

Similarly, for $z < 0$, we can add a contour C in the lower-half plane that contributes to zero to the integral, one can deform the contour to pick up the pole contribution. Hence, the integral is equal to the pole contribution at $k_z' = -(k_0^2 - k_x^2 - k_y^2)^{1/2}$ (see Figure 35.1). As such, the result for all z can be written as

$$\phi(x, y, z) = \frac{i}{2(2\pi)^2} \iint_{-\infty}^{\infty} dk_x dk_y \, \frac{e^{ik_x x + ik_y y + ik_z' |z|}}{k_z'}, \quad \text{all } z. \tag{35.1.10}$$

By the uniqueness of the solution to the partial differential equation (35.1.1) satisfying radiation condition at infinity, we can equate (35.1.2) and (35.1.10), yielding the identity

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi} \iint_{-\infty}^{\infty} dk_x dk_y \, \frac{e^{ik_x x + ik_y y + ik_z |z|}}{k_z}, \tag{35.1.11}$$

where $k_x^2 + k_y^2 + k_z^2 = k_0^2$, or $k_z = (k_0^2 - k_x^2 - k_y^2)^{1/2}$. The above is known as the ***Weyl identity*** (Weyl 1919). To ensure the radiation condition, we require that $\Im m[k_z] > 0$ and $\Re e[k_z] > 0$ over all values of k_x and k_y in the integration. Furthermore, Equation (35.1.11) could be interpreted as an integral summation of plane waves propagating in all directions, including evanescent waves. It is the plane-wave expansion (including evanescent wave) of a spherical wave.

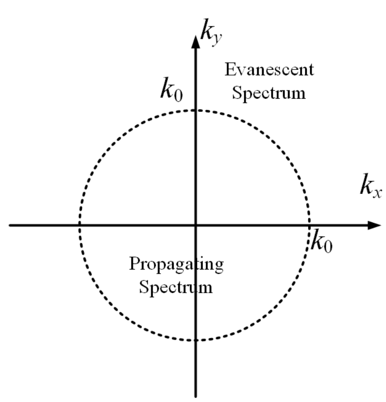

Figure 35.2: The wave is propagating for $\mathbf{k}_\rho$ vectors inside the disk, while the wave is evanescent for $\mathbf{k}_\rho$ outside the disk.

One can also interpret the above as a 2D surface integral in the Fourier space over the k_x and k_y plane or variables. When $k_x^2 + k_y^2 < k_0^2$, or inside a disk of radius k_0, the waves are propagating waves. But for contributions outside this disk, the waves are evanescent (see Figure 35.2). And the high Fourier (or spectral) components of the Fourier spectrum correspond to evanescent waves. Since high spectral components, which are related to the evanescent waves, are important for reconstructing the singularity of the Green's function.

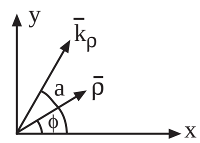

Figure 35.3: The $\mathbf{k}_\rho$ and the $\boldsymbol{\rho}$ vector on the xy plane.

Sommerfeld Identity

The Weyl identity has double integral, and hence, is more difficult to integrate numerically. Here, we shall derive the Sommerfeld identity which has only one integral. In (35.1.11), we can write $\mathbf{k}_\rho = \hat{x}k_\rho \cos\alpha + \hat{y}k_\rho \sin\alpha$, $\boldsymbol{\rho} = \hat{x}\rho\cos\phi + \hat{y}\rho\sin\phi$ (see Figure 35.3), and $dk_x dk_y = k_\rho dk_\rho\, d\alpha$. Then, $k_x x + k_y y = \mathbf{k}_\rho \cdot \boldsymbol{\rho} = k_\rho \cos(\alpha - \phi)$, and we have

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi} \int_0^\infty k_\rho dk_\rho \int_0^{2\pi} d\alpha \frac{e^{ik_\rho \rho\cos(\alpha-\phi)+ik_z|z|}}{k_z}, \tag{35.1.12}$$

where $k_z = (k_0^2 - k_x^2 - k_y^2)^{1/2} = (k_0^2 - k_\rho^2)^{1/2}$, where in cylindrical coordinates, in the $\mathbf{k}_\rho$-space, or the Fourier space, $k_\rho^2 = k_x^2 + k_y^2$. Then, using the integral identity for Bessel functions given by[4]

$$J_0(k_\rho \rho) = \frac{1}{2\pi} \int_0^{2\pi} d\alpha\, e^{ik_\rho \rho \cos(\alpha-\phi)}, \tag{35.1.13}$$

(35.1.12) becomes

$$\frac{e^{ik_0 r}}{r} = i \int_0^\infty dk_\rho\, \frac{k_\rho}{k_z} J_0(k_\rho \rho) e^{ik_z|z|}. \tag{35.1.14}$$

The above is also known as the ***Sommerfeld identity*** (Sommerfeld 1909 [207]; [191][p. 242]). Its physical interpretation is that a spherical wave can be expanded as an integral summation of conical waves or cylindrical waves in the ρ direction, times a plane wave in the z direction over all wave numbers k_ρ. This wave is evanescent in the $\pm z$ direction when $k_\rho > k_0$.

[4]See Chew [34], or Whitaker and Watson(1927) [206].

By using the fact that $J_0(k_\rho \rho) = 1/2[H_0^{(1)}(k_\rho \rho) + H_0^{(2)}(k_\rho \rho)]$, and the reflection formula that $H_0^{(1)}(e^{i\pi}x) = -H_0^{(2)}(x)$, a variation of the above identity can be derived as

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2}\int_{-\infty}^{\infty} dk_\rho \, \frac{k_\rho}{k_z} H_0^{(1)}(k_\rho \rho) e^{ik_z|z|}. \tag{35.1.15}$$

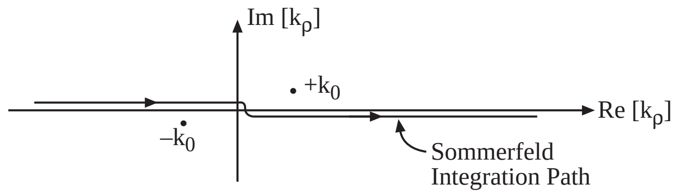

Figure 35.4: Sommerfeld integration path.

Since $H_0^{(1)}(x)$ has a logarithmic branch-point singularity at $x = 0$,[5] and $k_z = (k_0^2 - k_\rho^2)^{1/2}$ has algebraic branch-point singularities at $k_\rho = \pm k_0$, the integral in Equation (35.1.15) is undefined unless we stipulate also the path of integration. Hence, a path of integration adopted by Sommerfeld, which is even good for a lossless medium, is shown in Figure 35.4. Because of the manner in which we have selected the reflection formula for Hankel functions, i.e., $H_0^{(1)}(e^{i\pi}x) = -H_0^{(2)}(x)$, the path of integration should be above the logarithmic branch-point singularity at the origin. With this definition of the Sommerfeld integration, the integral is well defined even when there is no loss, i.e., when the bronach points $\pm k_0$ are on the real axis.

35.2 A Source on Top of a Layered Medium

It can be shown that plane waves reflecting from a layered medium can be decomposed into TE-type plane waves, where $E_z = 0$, $H_z \neq 0$, and TM-type plane waves, where $H_z = 0$, $E_z \neq 0$.[6] One also sees how the field due to a point source can be expanded into plane waves in Section 35.1.

In view of the above observations, when a point source is on top of a layered medium, it is then best to decompose its field in terms of waves of TE-type and TM-type. Then, the nonzero component of E_z characterizes TM waves, while the nonzero component of H_z characterizes TE waves. Hence, given a field, its TM and TE components can be extracted readily. Furthermore, if these TM and TE components are expanded in terms of plane waves, their propagations in a layered medium can be studied easily.

The problem of a vertical electric dipole on top of a half space was first solved by Sommerfeld (1909) [207] using Hertzian potentials, which are related to the z components of the

[5] $H_0^{(1)}(x) \sim \frac{2i}{\pi}\ln(x)$, see Chew [34][p. 14], or Abromawitz or Stegun [107].
[6] *Chew, Waves and Fields in Inhomogeneous Media* [34]; *Kong, Electromagnetic Wave Theory* [31].

electromagnetic field. The work is later generalized to layered media, as discussed in the literature. Later, Kong (1972) [208] suggested the use of the z components of the electromagnetic field instead of the Hertzian potentials.

35.2.1 Electric Dipole Fields–Spectral Expansion

The representation of a spherical wave in terms of plane waves can be done using Weyl identity or Sommerfeld identiy. Here, we will use Sommerfeld identity in anticipation of numerical integration, since only single integrals are involved. The **E** field in a homogeneous medium due to a point current source or a Hertzian dipole directed in the $\hat{\alpha}$ direction, $\mathbf{J} = \hat{\alpha} I\ell\,\delta(\mathbf{r})$, is derivable via the vector potential method or the dyadic Green's function approach. Then, using the dyadic Green's function approach, or the vector/scalar potential approach, the field due to a Hertzian dipole is given by

$$\mathbf{E}(\mathbf{r}) = i\omega\mu\left(\bar{\mathbf{I}} + \frac{\nabla\nabla}{k^2}\right)\cdot\hat{\alpha} I\ell\,\frac{e^{ikr}}{4\pi r}, \tag{35.2.1}$$

where $I\ell$ is the current moment and $k = \omega\sqrt{\mu\epsilon}$, the wave number of the homogeneous medium. Furthermore, from $\nabla \times \mathbf{E} = i\omega\mu\mathbf{H}$, the magnetic field due to a Hertzian dipole is given by

$$\mathbf{H}(\mathbf{r}) = \nabla \times \hat{\alpha} I\ell\,\frac{e^{ikr}}{4\pi r}. \tag{35.2.2}$$

With the above fields, their TM and TE components can be extracted easily.

(a) Vertical Electric Dipole (VED)

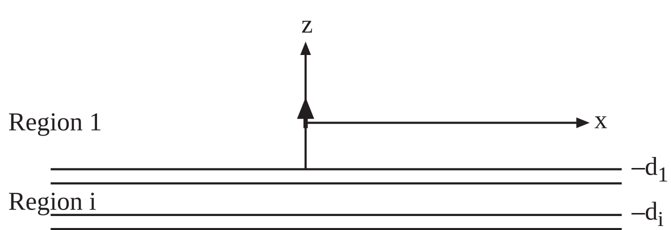

Figure 35.5: A vertical electric dipole over a layered medium.

A vertical electric dipole shown in Figure 35.5 has $\hat{\alpha} = \hat{z}$; hence, the TM component of the field is characterized by $E_z \neq 0$ or that

$$E_z = \frac{i\omega\mu I\ell}{4\pi k^2}\left(k^2 + \frac{\partial^2}{\partial z^2}\right)\frac{e^{ikr}}{r}, \tag{35.2.3}$$

and the TE component of the field is characterized by

$$H_z = 0, \tag{35.2.4}$$

implying the absence of the TE field.

Next, using the Sommerfeld identity (35.1.15) in the above, and after exchanging the order of integration and differentiation, we have[7]

$$E_z = \frac{-I\ell}{4\pi\omega\epsilon} \int\limits_0^\infty dk_\rho \, \frac{k_\rho^3}{k_z} J_0(k_\rho\rho) e^{ik_z|z|}, \tag{35.2.5}$$

after noting that $k_\rho^2 + k_z^2 = k^2$. Notice that now Equation (35.2.5) expands the z component of the electric field in terms of cylindrical waves in the ρ direction and a plane wave in the z direction. Since cylindrical waves actually are linear superpositions of plane waves, because we can work backward from (35.1.15) to (35.1.11) to see this. As such, the integrand in (35.2.5) in fact consists of a linear superposition of TM-type plane waves. The above is also the ***primary field*** generated by the source.

Consequently, for a VED on top of a stratified medium as shown, the downgoing plane wave from the point source will be reflected like TM waves with the generalized reflection coefficient $\tilde{R}_{12}^{TM}$. Hence, over a stratified medium, the field in region 1 can be written as

$$E_{1z} = \frac{-I\ell}{4\pi\omega\epsilon_1} \int_0^\infty dk_\rho \frac{k_\rho^3}{k_{1z}} J_0(k_\rho\rho) \left[e^{ik_{1z}|z|} + \tilde{R}_{12}^{TM} e^{ik_{1z}z + 2ik_{1z}d_1} \right], \tag{35.2.6}$$

where $k_{1z} = (k_1^2 - k_\rho^2)^{\frac{1}{2}}$, and $k_1^2 = \omega^2\mu_1\epsilon_1$, the wave number in region 1.

The phase-matching condition dictates that the transverse variation of the field in all the regions must be the same. Consequently, in the i-th region, the solution becomes

$$\epsilon_i E_{iz} = \frac{-I\ell}{4\pi\omega} \int_0^\infty dk_\rho \frac{k_\rho^3}{k_{1z}} J_0(k_\rho\rho) A_i \left[e^{-ik_{iz}z} + \tilde{R}_{i,i+1}^{TM} e^{ik_{iz}z + 2ik_{iz}d_i} \right]. \tag{35.2.7}$$

Notice that Equation (35.2.7) is now expressed in terms of $\epsilon_i E_{iz}$ because $\epsilon_i E_{iz}$ reflects and transmits like H_{iy}, the transverse component of the magnetic field or TM waves.[8] Therefore, $\tilde{R}_{i,i+1}^{TM}$ and A_i could be obtained using the methods discussed in *Chew, Waves and Fields in Inhomogeneous Media*.

This completes the derivation of the integral representation of the electric field everywhere in the stratified medium. These integrals are known as ***Sommerfeld integrals***. The case when the source is embedded in a layered medium can be derived similarly

[7]By using (35.1.15) in (35.2.3), the $\partial^2/\partial z^2$ operating on $e^{ik_z|z|}$ produces a Dirac delta function singularity. Detail discussion on this can be found in the chapter on dyadic Green's function in *Chew, Waves and Fields in Inhomogeneous Media* [34].

[8]See *Chew, Waves and Fields in Inhomogeneous Media* [34], p. 46, (2.1.6) and (2.1.7)

(b) Horizontal Electric Dipole (HED)

The HED is more complicated. Unlike the VED that excites only the TM waves, an HED will excite both TE and TM waves. For a horizontal electric dipole pointing in the x direction, $\hat{\alpha} = \hat{x}$; hence, (35.2.1) and (35.2.2) give the TM and the TE components as

$$E_z = \frac{iI\ell}{4\pi\omega\epsilon}\frac{\partial^2}{\partial z \partial x}\frac{e^{ikr}}{r}, \tag{35.2.8}$$

$$H_z = -\frac{I\ell}{4\pi}\frac{\partial}{\partial y}\frac{e^{ikr}}{r}. \tag{35.2.9}$$

Then, with the Sommerfeld identity (35.1.15), we can expand the above as

$$E_z = \pm\frac{iI\ell}{4\pi\omega\epsilon}\cos\phi \int_0^\infty dk_\rho\, k_\rho^2 J_1(k_\rho\rho)e^{ik_z|z|} \tag{35.2.10}$$

$$H_z = i\frac{I\ell}{4\pi}\sin\phi \int_0^\infty dk_\rho\, \frac{k_\rho^2}{k_z} J_1(k_\rho\rho)e^{ik_z|z|}. \tag{35.2.11}$$

Now, Equation (35.2.10) represents the wave expansion of the TM field, while (35.2.11) represents the wave expansion of the TE field. Observe that because E_z is odd about $z = 0$ in (35.2.10), the downgoing wave has an opposite sign from the upgoing wave. At this point, the above are just the primary field generated by the source.

On top of a stratified medium, the downgoing wave is reflected accordingly, depending on its wave type. Consequently, we have

$$E_{1z} = \frac{iI\ell}{4\pi\omega\epsilon_1}\cos\phi \int_0^\infty dk_\rho\, k_\rho^2 J_1(k_\rho\rho)\left[\pm e^{ik_{1z}|z|} - \tilde{R}_{12}^{TM} e^{ik_{1z}(z+2d_1)}\right], \tag{35.2.12}$$

$$H_{1z} = \frac{iI\ell}{4\pi}\sin\phi \int_0^\infty dk_\rho\, \frac{k_\rho^2}{k_{1z}} J_1(k_\rho\rho)\left[e^{ik_{1z}|z|} + \tilde{R}_{12}^{TE} e^{ik_{1z}(z+2d_1)}\right]. \tag{35.2.13}$$

Notice that the negative sign in front of $\tilde{R}_{12}^{TM}$ in (35.2.12) follows because the downgoing wave in the primary field has a negative sign.

35.3 Stationary Phase Method

Sommerfeld integrals are rather complex, and by themselves, they do not offer much physical insight into the physics of the field. To elucidate the physics, we can apply the stationary phase method to find approximations of these integrals when the frequency is high. In order to avoid having to work with special functions like Bessel functions, we convert the Sommerfeld integrals back to spectral integrals in the cartesian coordinates. We could have obtained the aforementioned integrals in cartesian coordinates were we to start with the Weyl identity instead of the Sommerfeld identity. To do the back conversion, we make use of the identity,

$$\frac{e^{ik_0 r}}{r} = \frac{i}{2\pi} \iint_{-\infty}^{\infty} dk_x dk_y \frac{e^{ik_x x + ik_y y + ik_z |z|}}{k_z} = i \int_0^{\infty} dk_\rho \frac{k_\rho}{k_z} J_0(k_\rho \rho) e^{ik_z |z|}. \tag{35.3.1}$$

We can just focus our attention on the reflected wave term in (35.2.6) and rewrie it in cartesian coordinates to get

$$\begin{aligned} E_{1z}^R &= \frac{-I\ell}{8\pi^2 \omega \epsilon_1} \iint_{-\infty}^{\infty} dk_x dk_y \frac{k_x^2 + k_y^2}{k_{1z}} R_{12}^{TM} e^{ik_x x + ik_y y + ik_{1z}(z+2d_1)} \\ &= \iint_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{1z}} F(k_x, k_y) e^{ik_x x + ik_y y + ik_{1z}(z+2d_1)} \end{aligned} \tag{35.3.2}$$

where

$$F(k_x, k_y) = \frac{-I\ell}{8\pi^2 \omega \epsilon_1} (k_x^2 + k_y^2) R_{12}^{TM}$$

In the above, $k_x^2 + k_y^2 + k_{1z}^2 = k_1^2$ is the dispersion relation satisfied by the plane wave in region 1. Also, R_{12}^{TM} is dependent on $k_{iz} = \sqrt{k_i^2 - k_x^2 - k_y^2}$ in cartesian coordinates, where $i = 1, 2$.

Now the problem reduces to finding the approximation of the following integral:

$$E_{1z}^R = \iint_{-\infty}^{\infty} dk_x dk_y \frac{1}{k_{1z}} F(k_x, k_y) e^{i\lambda h(k_x, k_y)} \tag{35.3.3}$$

where

$$\lambda h(k_x, k_y) = \left(k_x \frac{x}{r} + k_y \frac{y}{r} + k_{1z} \frac{z}{r} \right) r, \qquad g(k_x, k_y, \lambda) = e^{i\lambda h(k_x, k_y)} \tag{35.3.4}$$

The large parameter here is $\lambda = r$. For simplicity, we have set $d_1 = 0$ to begin with.

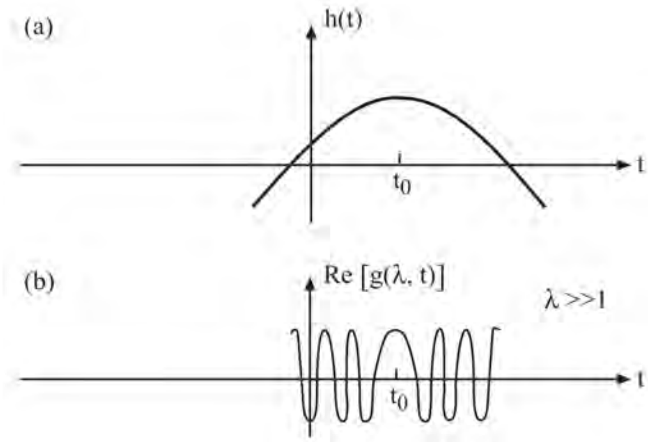

Figure 35.6: In this figure, t can represent k_x or k_y when one of them is varying. Around the stationary phase point, the function $h(t)$ is slowly varying. When $\lambda = r$ is large, the function $g(\lambda, k_x, k_y)$ is rapidly varying with respect to either k_x or k_y. Hence, most of the contributions to the integral comes from around the stationary phase point.

In the above, $e^{i\lambda h(k_x,k_y)}$ is a rapidly varying function of k_x and k_y when x, y, and z or $\lambda = r$ are large compared to wavelength.[9] In other words, a small change in k_x or k_y will cause a large change in the phase of the integrand, or the integrand will be a rapidly varying function of k_x and k_y. Due to the cancellation of the integral when one integrates a rapidly varying function, most of the contributions to the integral will come from around the stationary point of $h(k_x, k_y)$ or where the function is least slowly varying. Otherwise, the integrand is rapidly varying away from this point, and the integration will destructively cancel with each other, while around the stationary point, they will add constructively.

The stationary point in the k_x and k_y plane is found by setting the derivatives of $h(k_x, k_y)$ with respect to to k_x and k_y to zero. By so doing

$$\frac{\partial h}{\partial k_x} = \frac{x}{r} - \frac{k_x}{k_{1z}}\frac{z}{r} = 0, \qquad \frac{\partial h}{\partial k_y} = \frac{y}{r} - \frac{k_y}{k_{1z}}\frac{z}{r} = 0 \tag{35.3.5}$$

The above represents two equations from which the two unknowns, k_{xs} and k_{ys}, at the stationary phase point can be solved for. By expressing the above in spherical coordinates, $x = r\sin\theta\cos\phi$, $y = r\sin\theta\sin\phi$, $z = r\cos\theta$, the values of (k_{xs}, k_{ys}), that satisfy the above equations are

$$k_{xs} = k_1 \sin\theta\cos\phi, \quad k_{ys} = k_1 \sin\theta\sin\phi \tag{35.3.6}$$

with the corresponding $k_{1zs} = k_1 \cos\theta$.

When one integrates on the k_x and k_y plane, the dominant contribution to the integral will come from the point in the vicinity of (k_{xs}, k_{ys}). Assuming that $F(k_x, k_y)$ is slowly varying,

[9]The yardstick in wave physics is always wavelength. Large distance is also synonymous to increasing the frequency or reducing the wavelength.

we can equate $F(k_x, k_y)$ to a constant equal to its value at the stationary phase point, and say that

$$E_{1z}^R \simeq F(k_{xs}, k_{ys}) \iint_{-\infty}^{\infty} \frac{1}{k_{1z}} e^{ik_x x + ik_y y + ik_{1z} z} dk_x dk_y = 2\pi F(k_{xs}, k_{ys}) \frac{e^{ik_1 r}}{ir} \tag{35.3.7}$$

The above has two important physical interpretations.

(i) Even though a source is emanating plane waves in all directions in accordance to (35.1.11), at the observation point r far away from the source point, only one or few plane waves in the vicinity of the stationary phase point are important. They interfere with each other constructively to form a spherical wave that represents the ray connecting the source point to the observation point. Plane waves in other directions interfere with each other destructively, and are not important. That is the reason that the source point and the observation point is connected only by one ray, or one bundle of plane waves in the vicinity of the stationary phase point.

(ii) The function $F(k_x, k_y)$ could be a very complicated function like the reflection coefficient R^{TM}, but only its value at the stationary phase point matters. If we were to make $d_1 \neq 0$ again in the above analysis, then $r \rightarrow r_I = \sqrt{x^2 + y^2 + (z + 2d_1)^2}$. Due to the reflecting half-space, the source point has an image point as shown in Figure 35.7 This physical picture is shown in the figure where r_I now is the distance of the observation point to the image point. The stationary phase method extract a ray that emanates from the source point, bounces off the half-space, and the reflected ray reaches the observer modulated by the reflection coefficient R^{TM}. But the value of the reflection coefficient that matters is at the angle at which the incident ray impinges on the half-space.

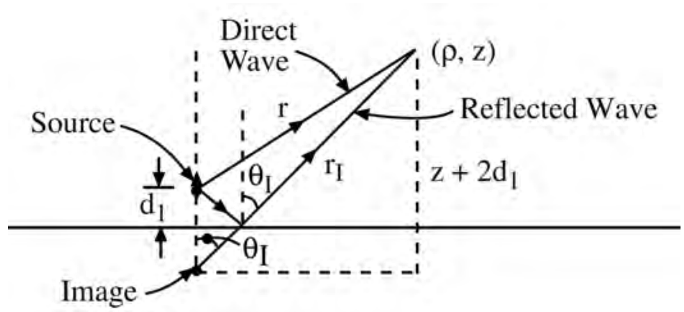

Figure 35.7: At high frequencies, the source point and the observation point are connected by a ray. The ray represents a bundle of plane waves that interfere constructively. This even true for a bundle of plane waves that reflect off an interface. So ray theory or ray optics prevails here, and the ray bounces off the interface according to the reflection coefficient of a plane wave impinging at the interface with θ_I.

35.4 Riemann Sheets and Branch Cuts

The Sommerfeld integrals will have integrands that are multi-value or double value. Proper book keeping is needed so that the evaluation of these integrals can be performed unambiguously. The function $k_z = (k_0^2 - k_\rho^2)^{1/2}$ in (35.1.14) and (35.1.15) are double-value functions because, in taking the square root of a number, two values are possible. In particular, k_z is a double-value function of k_ρ. Consequently, for every point on a complex k_ρ plane in Figure 35.4, there are two possible values of k_z. Therefore, the integral (35.1.10) is undefined unless we stipulate which of the two values of k_z is adopted in performing the integration.

A multivalue function is denoted on a complex plane with the help of ***Riemann sheets*** [34, 82]. For instance, a double-value function such as k_z is assigned two Riemann sheets to a single complex plane. On one of these Riemann sheets, k_z assumes a value just opposite in sign to the value on the other Riemann sheet. The correct sign for k_z is to pick the square root solution so that $\Im m(k_z) > 0$. This will ensure a decaying wave from the source.

35.5 Some Remarks

Even though we have arrived at the solutions of a point source on top of a layered medium by heuristic arguments of plane waves propagating through layered media, they can also be derived more rigorously. For example, Equation (35.2.6) can be arrived at by matching boundary conditions at every interface. The reason why a more heuristic argument is still valid is due to the completeness of Fourier transforms. It is best explained by putting a source over a half space and a scalar problem.

We can expand the scalar field in the upper region as

$$\Phi_1(x, y, z) = \iint_{-\infty}^{\infty} dk_x dk_y \tilde{\Phi}_1(k_x, k_y, z) e^{ik_x x + ik_y y} \tag{35.5.1}$$

and the scalar field in the lower region as

$$\Phi_2(x, y, z) = \iint_{-\infty}^{\infty} dk_x dk_y \tilde{\Phi}_2(k_x, k_y, z) e^{ik_x x + ik_y y} \tag{35.5.2}$$

If we require that the two fields be equal to each other at $z = 0$, then we have

$$\iint_{-\infty}^{\infty} dk_x dk_y \tilde{\Phi}_1(k_x, k_y, z = 0) e^{ik_x x + ik_y y} = \iint_{-\infty}^{\infty} dk_x dk_y \tilde{\Phi}_2(k_x, k_y, z = 0) e^{ik_x x + ik_y y} \tag{35.5.3}$$

In order to remove the integral, and replace it with a simple scalar problem, one has to impose the above equation for all x and y. Then the completeness of Fourier transform implies that[10]

$$\tilde{\Phi}_1(k_x, k_y, z = 0) = \tilde{\Phi}_2(k_x, k_y, z = 0) \tag{35.5.4}$$

[10]Or that we can perform a Fourier inversion on the above integrals.

The above equation is much simpler than that in (35.5.3). In other words, due to the completeness of Fourier transform, one can match a boundary condition spectral-component by spectral-component. If the boundary condition is matched for all spectral components, than (35.5.3) is also true.